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Successive Approximations for a Time Dependent Scattering Process

C. V. PAO

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607

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1. INTRODUCTION

In a recent paper [1], Bellman has discussed the following system of partial differential equations

$$x_t - x_r = Ax + Dy, \qquad y_t + y_r = Bx + Cy, \tag{1.1}$$

under the boundary condition

$$x(l, t) = c, \qquad y(0, t) = d, \tag{1.2}$$

and the initial condition

$$x(r, 0) = y(r, 0) = 0, \tag{1.3}$$

where $x = x(r, t)$, $y = y(r, t)$, c, d are vectors in the n-dimensional Euclidean space R^n and A, B, C, D are constant $n \times n$ matrices. The problem (1.1)–(1.3) describes an idealized transport process of n different types of particles moving in either direction along a line of finite lengt l. Using Laplace transform, Bellman investigated the asymptotic behavior of the solution and its relationship with the steady state solution. In this paper, we are concerned with the questions of the well-posed problem and the method of construction of a solution for the following more general system of initial boundary value problem:

$$\begin{aligned} x_t - x_r &= A(r, t)x + D(r, t)y + p(r, t) \\ &\qquad\qquad (0 < x < l, t > 0) \qquad (2.1 \\ y_t + y_r &= B(r, t)x + C(r, t)y + q(r, t) \end{aligned}$$

$$x(l, t) = c, y(0, t) = d \qquad (t > 0) \tag{2.2}$$

$$x(r, 0) = \phi(r), y(r, 0) = \psi(r) \qquad (0 < r < l). \tag{2.3}$$

In the above system, the matrices A, B, C, D can be functions of position r as well as time t. The additional terms p, q in (2.1) and ϕ, ψ in (2.3) include possible external sources and initial fluxes in the transport medium. The boundary condition (2.2) means that incident fluxes of constant intensity are applied at both ends of the line. As in the usual sense, the well-posed problem considered in this paper consists of (a) the existence of a solution, (b) the uniqueness problem and, (c) the continuous dependence of the solution on the external sources and the initial-boundary data. (For a derivation of the problem the reader is referred to [1] or [2, Chapter 7]. Some discussion on the corresponding steady-state problem can be found in [3] and [4]).

In order to insure the existence of a solution we assume that the matrices A, B, C, D and the vectors p, q, ϕ, ψ are essentially bounded on $[0, l] \times [0, T]$ for every finite value of T. This includes the most interesting case where A, B, C, D are piecewise continuous (e.g., see [3]). Our basic approach in proving the existence and uniqueness problem is by the method of successive approximations. An essential contribution of this approach is that it leads to a recursion formula for the determination of the solution as well as error estimates for the approximations. In the recursion formula, the spatial variable r is fixed through the process of iterations. This property is especially useful since one is often interested only in the reflected fluxes $x(0, t), y(l, t)$ at the ends of the line. Furthermore, since our recursion formula involves only straightforward integrations just as in the initial-value problems of ordinary differential equations, numerical results for the approximations can be handled by a digital computer. Thus our approach to the problem provides both analytic results and computational significance.

2. Approach to the Problem

In this section we describe our approach to the problem (2.1)–(2.3) by the method of successive approximations. This approach is similar to that given in [5] for the treatment of multivelocity neutron transport problems. The process of approximations is as follows: Starting from a given pair of functions $(x^{(0)}, y^{(0)})$ we construct a sequence $\{x^{(k)}, y^{(k)}\}$ from the following interrelated (but not coupled) systems

$$\left.\begin{aligned} x_t^{(k)} - x_r^{(k)} &= Ax^{(k-1)} + Dy^{(k-1)} + p && (0 \leqslant r < l, t > 0)\\ x^{(k)}(l, t) &= c && (t > 0)\\ x^{(k)}(r, 0) &= \phi(r) && (0 \leqslant r < l)\end{aligned}\right\} k = 1, 2, \ldots, \tag{3.1}$$

$$\left.\begin{aligned} y_t^{(k)} + y_r^{(k)} &= Bx^{(k-1)} + Cy^{(k-1)} + q && (0 < r \leqslant l, t > 0)\\ y^{(k)}(0, t) &= d && (t > 0)\\ y^{(k)}(r, 0) &= \psi(r) && (0 < r \leqslant l)\end{aligned}\right\} k = 1, 2, \ldots, \tag{3.2}$$

The construction of the sequence $\{x^{(k)}, y^{(k)}\}$ is clear since for each $k = 1, 2,...,$ the right side of the equations in (3.1) and (3.2) are both known functions. In fact, as we will show in the following section (see Lemma 3.3) that the systems (3.1) and (3.2) can be solved for $x^{(k)}$ and $y^{(k)}$, respectively, to yield the recursion formula

$$\left.\begin{aligned} x^{(k)}(r, t) &= \hat{\phi}(r + t) + \int_0^t A(r + t - \tau, \tau)\, x^{(k-1)}(r + t - \tau, \tau)\, d\tau \\ &\quad + \int_0^t D(r + t - \tau, \tau)\, y^{(k-1)}(r + t - \tau, \tau)\, d\tau \\ &\quad + \int_0^t p(r + t - \tau, \tau)\, d\tau \qquad (0 \leqslant r \leqslant l, t \geqslant 0) \\ y^{(k)}(r, t) &= \hat{\psi}(r - t) + \int_0^t B(r - t + \tau, \tau)\, x^{(k-1)}(r - t + \tau, \tau)\, d\tau \\ &\quad + \int_0^t C(r - t + \tau, \tau)\, y^{(k-1)}(r - t + \tau, \tau)\, d\tau \\ &\quad + \int_0^t q(r - t + \tau, \tau)\, d\tau \qquad (0 \leqslant r \leqslant l, t \geqslant 0) \end{aligned}\right\} k = 1, 2,..., \tag{3.3}$$

where $\hat{\phi}$, $\hat{\psi}$ are defined by

$$\hat{\phi}(r) = \begin{cases} \phi(r) & \text{if} \quad r < l, \\ c & \text{if} \quad r \geqslant l, \end{cases} \qquad \hat{\psi}(r) = \begin{cases} \psi(r) & \text{if} \quad r > 0, \\ d & \text{if} \quad r \leqslant 0. \end{cases} \tag{3.4}$$

In the recursion formula it is also defined that

$$\begin{aligned} A(r, t) = B(r, t) = C(r, t) = D(r, t) = 0, \\ p(r, t) = q(r, t) = 0 \qquad \text{when} \qquad r \notin [0, l]. \end{aligned} \tag{3.5}$$

Hence our crucial problem is to show that the sequence $\{x^{(k)}, y^{(k)}\}$ given by (3.3) converges to a unique solution of the problem (2.1)–(2.3). To accomplish this, we first transform the problem (2.1)–(2.3) by letting $u = e^{-\lambda t}x$, $v = e^{-\lambda t}y$ to obtain the transformed system

$$\begin{aligned} u_t - u_r + \lambda u = Au + Dv + e^{-\lambda t}p \\ v_t + v_r + \lambda v = Bu + Cv + e^{-\lambda t}q \end{aligned} \qquad (0 < r < l, t > 0), \tag{3.6}$$

$$u(l, t) = ce^{-\lambda t}, \qquad v(0, t) = de^{-\lambda t} \qquad (t > 0), \tag{3.7}$$

$$u(r, 0) = \phi(r), \qquad v(r, 0) = \psi(r) \qquad (0 < r < l), \tag{3.8}$$

where $\lambda \geqslant 0$ is a constant to be chosen. Similarly, by letting $u^{(k)} = e^{-\lambda t}x^{(k)}$,

$v^{(k)} = e^{-\lambda t} y^{(k)}$ we transform the corresponding systems (3.1) and (3.2) into the respective form

$$\left.\begin{aligned} u_t^{(k)} - u_r^{(k)} + \lambda u^{(k)} &= Au^{(k-1)} + Dv^{(k-1)} + e^{-\lambda t} p \\ u^{(k)}(l, t) &= ce^{-\lambda t} \\ u^{(k)}(r, 0) &= \phi(r) \end{aligned}\right\} k = 1, 2,..., \tag{3.9}$$

$$\left.\begin{aligned} v_t^{(k)} + v_r^{(k)} + \lambda v^{(k)} &= Bu^{(k-1)} + Cv^{(k-1)} + e^{-\lambda t} q \\ v^{(k)}(0, t) &= de^{-\lambda t} \\ v^{(k)}(r, 0) &= \psi(r) \end{aligned}\right\} k = 1, 2,.... \tag{3.10}$$

Thus to show the convergence of $\{x^{(k)}, y^{(k)}\}$ to a unique solution of (2.1)–(2.3) it suffices to show the convergence of $\{u^{(k)}, v^{(k)}\}$ to a unique solution of (3.6)–(3.8). For this purpose, we formulate the problem (3.6)–(3.8) as an operator equation in a suitable function space.

Let $\Omega = (0, l) \times (0, T]$ for some finite but arbitrary value of T and let $\mathscr{C}(\Omega)$ be the set of all continuous vector functions $u = (u_1 ,..., u_n)$ on Ω. As usual, we denote by $L^2(\Omega)$ the space of all square Lebesgue integrable vector-functions in Ω. The inner product between $u = (u_1 ,..., u_n)$ and $v = (v_1 ,..., v_n)$ and the norm of u are given, respectively, by

$$\langle u, v\rangle = \int_\Omega u(r, t) \cdot v(r, t)\, dr\, dt = \int_0^T \int_0^l (u_1 v_1 + \cdots + u_n v_n)\, dr\, dt$$

$$\| u \| = \left(\int_\Omega | u(r, t)|^2\, dr\, dt\right)^{1/2},$$

where $u \cdot v$ and $| u |$ denote the usual Euclidean inner product and norm in R^n. We next let $\mathscr{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ be the product space of $L^2(\Omega)$ equipped with the inner product

$$\langle U, U'\rangle = \int_\Omega (u \cdot u' + v \cdot v')\, dr\, dt = \int_0^T \int_0^l \sum_{i=1}^n (u_i u_i' + v_i v_i')\, dr\, dt$$

and norm $\| U \| = \langle U, U\rangle^{1/2}$, where $U = (u, v)$, $U' = (u', v')$ are in $\mathscr{L}^2(\Omega)$. Since it seems there is no confusion we use the same inner product and norm notations for both $L^2(\Omega)$ and $\mathscr{L}^2(\Omega)$.

Define operators $\mathscr{A}_i$, F_i, $i = 1, 2$, by

$$\begin{aligned} \mathscr{A}_1 u &= u_t - u_r + \lambda u \qquad (u \in D(\mathscr{A}_1)), \\ \mathscr{A}_2 v &= v_t + v_r + \lambda v \qquad (v \in D(\mathscr{A}_2)), \end{aligned} \tag{3.11}$$

$$\left.\begin{aligned} (F_1(u, v))(r, t) &= A(r, t)\, u(r, t) + D(r, t)\, v(r, t) + e^{-\lambda t} p(r, t), \\ (F_2(u, v))(r, t) &= B(r, t)\, u(r, t) + C(r, t)\, v(r, t) + e^{-\lambda t} q(r, t) \end{aligned}\right. \quad (u, v \in L^2(\Omega)), \tag{3.12}$$

where $D(\mathscr{A}_i)$ is the domain of $\mathscr{A}_i$ given by

$$\begin{aligned} D(\mathscr{A}_1) &= \{u \in \mathscr{C}(\Omega);\ \mathscr{A}_1 u \in L^2(\Omega),\ u(l, t) = ce^{-\lambda t},\ u(r, 0) = \phi(r)\}, \\ D(\mathscr{A}_2) &= \{v \in \mathscr{C}(\Omega);\ \mathscr{A}_2 v \in L^2(\Omega),\ v(0, t) = de^{-\lambda t},\ v(r, 0) = \psi(r)\}. \end{aligned} \tag{3.13}$$

Then for each $i = 1, 2$, $\mathscr{A}_i$ is an operator with domain and range (denoted by $R(\mathscr{A}_i)$) both in $L^2(\Omega)$ while F_i is an operator defined on $\mathscr{L}^2(\Omega)$ into $L^2(\Omega)$. In order to insure the existence of a solution in $\mathscr{L}^2(\Omega)$ we extend the operator $\mathscr{A}_i$ to its closure $\bar{\mathscr{A}}_i$. The definition of $\bar{\mathscr{A}}_i$ is that if $\{w^{(k)}\}$ is a sequence in $D(\mathscr{A}_i)$ such that $w^{(k)} \to w$ and $\mathscr{A} w^{(k)} \to w^*$ as $k \to \infty$ then $w \in D(\bar{\mathscr{A}}_i)$ and $\bar{\mathscr{A}}_i w = w^*$. With this definition the system (3.6)–(3.8) may be written as a coupled operator equation in the form

$$\begin{aligned} \bar{\mathscr{A}}_1 u &= F_1(u, v) \\ \bar{\mathscr{A}}_2 v &= F_2(u, v) \end{aligned} \qquad (u \in D(\bar{\mathscr{A}}_1),\ v \in D(\bar{\mathscr{A}}_2)). \tag{3.14}$$

The requirement of $u \in D(\bar{\mathscr{A}}_1)$, $v \in D(\bar{\mathscr{A}}_2)$ in Eq. (3.14) insures that the boundary and initial conditions (3.7) and (3.8) are satisfied by the solution of the equation. We next define operators $\mathscr{A}$, $\mathscr{F}$ by

$$\begin{aligned} \mathscr{A} U &= (\mathscr{A}_1 u, \mathscr{A}_2 v) \qquad && (U = (u, v) \in D(\mathscr{A})), \\ \mathscr{F}(U) &= (F_1(u, v), F_2(u, v)) \qquad && (U = (u, v) \in \mathscr{L}^2(\Omega)), \end{aligned} \tag{3.15}$$

where the domain $D(\mathscr{A})$ of $\mathscr{A}$ is given by $D(\mathscr{A}) = D(\mathscr{A}_1) \times D(\mathscr{A}_2)$. Then the coupled equations in (3.14) becomes the single operator equation

$$\mathscr{A} U = \mathscr{F}(U) \qquad (U \in D(\mathscr{A})) \tag{3.16}$$

in the Hilbert space $\mathscr{L}^2(\Omega)$. Notice that $\mathscr{A}$ is an operator with domain $D(\mathscr{A})$ and range $R(\mathscr{A})$ both in $\mathscr{L}^2(\Omega)$ and $\mathscr{F}$ is an operator defined on the whole space $\mathscr{L}^2(\Omega)$ into itself. In view of the above formulation, the systems (3.9) and (3.10) are equivalent to the respective operator equations

$$\begin{aligned} \bar{\mathscr{A}}_1 u^{(k)} &= F_1(u^{(k-1)}, v^{(k-1)}) \qquad && (u^{(k)} \in D(\bar{\mathscr{A}}_1)), \\ \bar{\mathscr{A}}_2 v^{(k)} &= F_2(u^{(k-1)}, v^{(k-1)}) \qquad && (v^{(k)} \in D(\bar{\mathscr{A}}_2)), \end{aligned} \tag{3.17}$$

which may be combined as the single operator equation

$$\mathscr{A} U^{(k)} = \mathscr{F}(U^{(k-1)}) \qquad (U^{(k)} \in D(\mathscr{A})) \tag{3.18}$$

in the space $\mathscr{L}^2(\Omega)$. Hence our problem is reduced to show that the sequence $\{U^{(k)}\}$ converges to a unique solution of (3.16). We prove this in the following section.

3. Convergence of the Approximations

Before proving the convergence of the sequence $\{u^{(k)}, v^{(k)}\}$ we prepare a series of lemmas. The first two lemmas give some properties of the operators $\mathscr{A}_i$ and $\mathscr{A}$.

Lemma 3.1. *For any real number* λ,

$$\begin{aligned} \langle \mathscr{A}_1 u - \mathscr{A}_1 u', u - u'\rangle &\geqslant \lambda \| u - u' \|^2 \qquad (u, u' \in D(\mathscr{A}_1)), \\ \langle \mathscr{A}_2 v - \mathscr{A}_2 v', v - v'\rangle &\geqslant \lambda \| v - v' \|^2 \qquad (v, v' \in D(\mathscr{A}_2)). \end{aligned} \tag{4.1}$$

If, in addition, $\lambda > 0$ *then for each* $i = 1, 2$, *the inverse operator* $\mathscr{A}_i^{-1}$ *exists and*

$$\| \mathscr{A}_i^{-1} w - \mathscr{A}_i^{-1} w' \| \leqslant \lambda^{-1} \| w - w' \| \qquad (w, w' \in R(\mathscr{A}_i)). \tag{4.2}$$

Proof. We first show the case for $\mathscr{A}_1$. Let $u, u' \in D(\mathscr{A}_1)$ and let $\tilde{u} = u - u'$. Then by definition, $\mathscr{A}_1 u - \mathscr{A}_1 u' = \tilde{u}_t - \tilde{u}_r + \lambda \tilde{u}$ and thus by integration,

$$\begin{aligned} \langle \mathscr{A}_1 u - \mathscr{A}_1 u', u - u'\rangle &= \int_0^T \int_0^l [(\tilde{u}_t - \tilde{u}_r + \lambda \tilde{u}) \cdot \tilde{u}]\, dr\, dt \\ &= \tfrac{1}{2} \int_0^l [| \tilde{u}(r, T)|^2 - | \tilde{u}(r, 0)|^2]\, dr \\ &\quad - \tfrac{1}{2} \int_0^T [|\tilde{u}(l, t)|^2 - | \tilde{u}(0, t)|^2]\, dt + \lambda \| \tilde{u} \|^2. \end{aligned} \tag{4.3}$$

Since $u, u' \in D(\mathscr{A}_1)$, the boundary and initial conditions (3.7) and (3.8) imply that $\tilde{u}(r, 0) = \phi(r) - \phi(r) = 0$ and $\tilde{u}(l, t) = ce^{-\lambda t} - ce^{-\lambda t} = 0$. It follows from (4.3) that

$$\langle \mathscr{A}_1 u - \mathscr{A}_1 u', u - u'\rangle \geqslant \lambda \| \tilde{u} \|^2 = \lambda \| u - u' \|^2,$$

which proves (4.1) for $\mathscr{A}_1$. To show the case for $\mathscr{A}_2$ we observe from the definition of $\mathscr{A}_2$ that

$$\begin{aligned} \langle \mathscr{A}_2 v - \mathscr{A}_2 v', v - v'\rangle &= \int_0^T \int_0^l [(\tilde{v}_t + \tilde{v}_r + \lambda \tilde{v}) \cdot \tilde{v}]\, dr\, dt \\ &= \tfrac{1}{2} \int_0^l [| \tilde{v}(r, T)|^2 - | \tilde{v}(r, 0)|^2]\, dr \\ &\quad + \tfrac{1}{2} \int_0^T [| \tilde{v}(l, t)|^2 - | \tilde{v}(0, t)|^2]\, dt + \lambda \| \tilde{v} \|^2, \end{aligned} \tag{4.4}$$

where $\tilde{v} = v - v'$. Using the second boundary and initial conditions for

v, v' we have $\tilde{v}(r, 0) = \tilde{v}(0, t) = 0$. Hence the inequality for $\mathscr{A}_2$ in (4.1) follows immediately from (4.4). In view of the relations in (4.1) we obtain, for $u, u' \in D(\mathscr{A}_i)$, $i = 1, 2$,

$$\| u - u' \| \| \mathscr{A}_i u - \mathscr{A}_i u' \| \geqslant \langle \mathscr{A}_i u - \mathscr{A}_i u', u - u' \rangle \geqslant \lambda \| u - u' \|^2. \tag{4.5}$$

The above relation implies that

$$\| \mathscr{A}_i u - \mathscr{A}_i u' \| \geqslant \lambda \| u - u' \| \qquad (u, u' \in D(\mathscr{A}_i)). \tag{4.6}$$

Hence the existence of $\mathscr{A}_i^{-1}$ and the inequality (4.2) follow directly from (4.6) by letting $w = \mathscr{A}_i u$, $w' = \mathscr{A}_i u'$. This completes the proof of the lemma.

LEMMA 3.2. *For any real* λ,

$$\langle \mathscr{A} U - \mathscr{A} U', U - U' \rangle \geqslant \lambda \| U - U' \|^2 \qquad (U, U' \in D(\mathscr{A})). \tag{4.7}$$

Moreover, if $\lambda > 0$ *then* $\mathscr{A}^{-1}$ *exists and*

$$\| \mathscr{A}^{-1} W - \mathscr{A}^{-1} W' \| \leqslant \lambda^{-1} \| W - W' \| \qquad (W, W' \in R(\mathscr{A})). \tag{4.8}$$

Proof. Let $U = (u, v)$, $U' = (u', v')$ be in $D(\mathscr{A})$. Then $u, u' \in D(\mathscr{A}_1)$, $v, v' \in D(\mathscr{A}_2)$ and (4.7) is equivalent to

$$\langle \bar{\mathscr{A}}_1 u - \bar{\mathscr{A}}_1 u', u - u' \rangle + \langle \bar{\mathscr{A}}_2 v - \bar{\mathscr{A}}_2 v', v - v' \rangle \geqslant \lambda(\| u - u' \|^2 + \| v - v' \|^2). \tag{4.9}$$

Thus it suffices to show that the inequalities in (4.1) hold for $\bar{\mathscr{A}}_1$, $\bar{\mathscr{A}}_2$, respectively. This is trivially true if $u, u' \in D(\mathscr{A}_1)$ and $v, v' \in D(\mathscr{A}_2)$ since $\bar{\mathscr{A}}_i$ is an extension of $\mathscr{A}_i$, $i = 1, 2$. For arbitrary $u, u' \in D(\bar{\mathscr{A}}_1)$ we can find, in view of the definition of $\bar{\mathscr{A}}_1$, sequences $\{w^{(k)}\}$, $\{z^{(k)}\}$ such that $w^{(k)} \to u$, $\mathscr{A}_1 w^{(k)} \to \bar{\mathscr{A}}_1 u$ and $z^{(k)} \to u'$, $\mathscr{A}_1 z^{(k)} \to \bar{\mathscr{A}}_1 u'$ as $k \to \infty$. But for each $k = 1, 2, \ldots$,

$$\langle \mathscr{A}_1 w^{(k)} - \mathscr{A}_1 z^{(k)}, w^{(k)} - z^{(k)} \rangle \geqslant \lambda \| w^{(k)} - z^{(k)} \|^2.$$

We have, by letting $k \to \infty$,

$$\langle \bar{\mathscr{A}}_1 u - \bar{\mathscr{A}}_1 u', u - u' \rangle \geqslant \lambda \| u - u' \|^2. \tag{4.10}$$

In the same manner,

$$\langle \bar{\mathscr{A}}_2 v - \bar{\mathscr{A}}_2 v', v - v' \rangle \geqslant \lambda \| v - v' \|^2. \tag{4.11}$$

Addition of (4.10) and (4.11) leads to the inequality (4.9) which proves the first part of the lemma. As in the proof of Lemma 3.1 we obtain from (4.7),

$$\| U - U' \| \| \mathscr{A}U - \mathscr{A}U' \| \geqslant \langle \mathscr{A}U - \mathscr{A}U', U - U' \rangle \geqslant \lambda \| U - U' \|^2. \tag{4.12}$$

The existence of $\mathscr{A}^{-1}$ and the inequality (4.8) follow from (4.12).

Our next lemma is concerned with the solution of the following two systems:

$$u_t - u_r + \lambda u = g, \qquad u(l, t) = ce^{-\lambda t}, \qquad u(r, 0) = \phi(r), \tag{4.13}$$

$$v_t + v_r + \lambda v = h, \qquad v(0, t) = de^{-\lambda t}, \qquad v(r, 0) = \psi(r), \tag{4.14}$$

where g, h are given functions in $L^2(\Omega)$. In the form of operator equations these systems are equivalent to

$$\begin{aligned} \bar{\mathscr{A}}_1 u &= g \qquad (u \in D(\bar{\mathscr{A}}_1)), \\ \bar{\mathscr{A}}_2 v &= h \qquad (v \in D(\bar{\mathscr{A}}_2)). \end{aligned} \tag{4.15}$$

Define

$$g(r, t) = 0 \quad \text{if} \quad r > l \qquad \text{and} \qquad h(r, t) = 0 \quad \text{if} \quad r < 0. \tag{4.16}$$

Then we have

LEMMA 3.3. *For any $g, h \in L^2(\Omega)$ there exist unique $u \in D(\bar{\mathscr{A}}_1)$, $v \in D(\bar{\mathscr{A}}_2)$ such that $\bar{\mathscr{A}}_1 u = g$, $\bar{\mathscr{A}}_2 v = h$. Furthermore the solutions u, v are given, respectively, by*

$$u(r, t) = e^{-\lambda t}\hat{\phi}(r + t) + \int_0^t e^{-\lambda(t-\tau)} g(r + t - \tau, \tau)\, d\tau \qquad (0 \leqslant r \leqslant l, t \geqslant 0), \tag{4.17}$$

$$v(r, t) = e^{-\lambda t}\hat{\psi}(r - t) + \int_0^t e^{-\lambda(t-\tau)} h(r - t + \tau, \tau)\, d\tau \qquad (0 \leqslant r \leqslant l, t \geqslant 0), \tag{4.18}$$

where $\hat{\phi}$, $\hat{\psi}$ are defined by (3.4).

Proof. We first solve the problem (4.13). Consider the case where $g \in \mathscr{C}(\Omega)$. Then by letting $r' = -r - t$, $t' = t$ the equation in (4.13) reduces to

$$\frac{du}{dt'}(-r' - t', t') + \lambda u(-r' - t', t') = g(-r' - t', t').$$

Multiplication by $e^{\lambda t'}$ and integration from 0 to t' lead to

$$e^{\lambda t'} u(-r' - t', t') - u(-r', 0) = \int_0^{t'} e^{\lambda\tau} g(-r' - \tau, \tau)\, d\tau.$$

Replacing r' by $-r - t$ and t' by t and using the initial condition (4.13) we obtain the formula (4.17) with $\hat{\phi}(r + t) = \phi(r + t)$. The above derivation shows that u satisfies the equation and the initial condition in (4.13). To show that u satisfies the boundary condition we observe from the definition (4.16) that $g(l + t - \tau, \tau) = 0$ for all $\tau \in (0, t)$. Thus the integral term in (4.17) vanishes at $r = l$. This together with the definition of $\hat{\phi}$ imply that

$$u(l, t) = e^{-\lambda t}\hat{\phi}(l + t) = ce^{-\lambda t} \qquad \text{for all } t > 0.$$

By Lemma 3.1 we conclude that for each $g \in \mathscr{C}(\Omega)$ there exists a unique $u \in D(\mathscr{A}_1)$ given by (4.17) such that $\mathscr{A}_1 u = g$. Notice that if g is piecewise continuous on Ω, so is u.

For $g \in L^2(\Omega)$ we choose a sequence $\{g^{(k)}\}$ in $\mathscr{C}(\Omega)$ such that $g^{(k)} \to g$ as $k \to \infty$. This is possible since $\mathscr{C}(\Omega)$ is dense in $L^2(\Omega)$. From the result just proved there exists a sequence $\{u^{(k)}\}$ in $D(\mathscr{A}_1)$ such that $\mathscr{A}_1 u^{(k)} = g^{(k)}$ for each $k = 1, 2,\ldots$. In view of Lemma 3.1, $u^{(k)} = \mathscr{A}_1^{-1} g^{(k)}$ and for any integers k, l,

$$\| u^{(k)} - u^{(l)} \| = \| \mathscr{A}_1^{-1} g^{(k)} - \mathscr{A}_1^{-1} g^{(l)} \| \leqslant \lambda^{-1} \| g^{(k)} - g^{(l)} \| .$$

Therefore $\{u^{(k)}\}$ is a Cauchy sequence and thus converges to some $u \in L^2(\Omega)$. This together with $\mathscr{A}_1 u^{(k)} \to g$ imply that $u \in D(\bar{\mathscr{A}_1})$ and $\bar{\mathscr{A}_1} u = g$. Following the same process except with the change of variables $r' = r - t$, $t' = t$ it is easily shown that the solution v for the problem (4.14) is given by the formula (4.18). This proves the lemma.

The implication of Lemma 3.3 is that if we let

$$\begin{aligned} g(r, t) &= (F_1(u^{(k-1)}, v^{(k-1)}))\,(r, t),\\ h(r, t) &= (F_2(u^{(k-1)}, v^{(k-1)}))\,(r, t), \end{aligned}$$

and use the definition (3.5) for r outside of $[0, l]$ then g, h satisfy the condition (4.16) for each $k = 1, 2,\ldots$, and thus the solution $u^{(k)}$ for the system (3.9) and $v^{(k)}$ for (3.10) are given, respectively, by

$$\begin{aligned} u^{(k)}(r, t) &= e^{-\lambda t}\hat{\phi}(r + t) + \int_0^t e^{-\lambda(t-\tau)}(F_1(u^{(k-1)}, v^{(k-1)}))\,(r + t - \tau, \tau)\, d\tau\\ v^{(k)}(r, t) &= e^{-\lambda t}\hat{\psi}(r - t) + \int_0^t e^{-\lambda(t-\tau)}(F_2(u^{(k-1)}, v^{(k-1)}))\,(r - t + \tau, \tau)\, d\tau \end{aligned}$$
$$k = 1, 2,\ldots, \qquad (4.19)$$

If the matrices A, B, C, D and the vectors p, q, ϕ, ψ are continuous or piecewise continuous then so are the functions g, h, and in this case the pair $(u^{(k)}, v^{(k)})$ given by (4.19) satisfies the problems (3.9), (3.10) in the classical sense.

For the sake of obtaining an explicit error estimate for the approximations $(u^{(k)}, v^{(k)})$ given by (4.19) we prepare one more lemma in which the constant M is defined by

$$M = \max_{1 \leqslant i \leqslant 4}\{\operatorname*{ess-sup}_{(r,t)\in\Omega} \Lambda_j(r, t)\}, \tag{4.20}$$

where Λ_j, $j = 1, 2, 3, 4$, are the largest eigenvalues of the matrices A^TA, B^TB, C^TC, D^TD, respectively.

LEMMA 3.4. *For any* $W, W' \in \mathscr{L}^2(\Omega)$,

$$\| \mathscr{F}(W) - \mathscr{F}(W')\| \leqslant K \| W - W' \|, \tag{4.21}$$

where $K = 2M^{1/2}$.

Proof. Let $W = (w, v)$, $W' = (w', v')$ and let $u = w - w'$, $u^* = v - v'$. Then by the definition of F_1, F_2,

$$\begin{aligned} \|F_1(W) - F_1(W')\| &= \| Au + Du^* \| \leqslant \| Au \| + \| Du^* \|, \\ \|F_2(W) - F_2(W')\| &= \| Bu + Cu^* \| \leqslant \| Bu \| + \| Cu^* \|. \end{aligned} \tag{4.22}$$

Since

$$(Au)\cdot(Au) = (A^TAu)\cdot u \leqslant \Lambda_1 \mid u \mid^2$$

we have

$$\| Au \|^2 = \int_\Omega (Au)\cdot(Au)\, dr\, dt \leqslant M \| u \|^2 \qquad (u \in L^2(\Omega)).$$

Similarly,

$$\| Bu \|^2 \leqslant M \| u \|^2, \qquad \| Cu^* \|^2 \leqslant M \| u^* \|^2, \qquad \| Du^* \|^2 \leqslant M \| u^* \|^2.$$

It follows from (4.22) that for each $i = 1, 2$,

$$\|F_i(W) - F_i(W')\| \leqslant M^{1/2}(\| u \| + \| u^* \|) \leqslant (2M)^{1/2} \| W - W' \|. \tag{4.23}$$

By the definition of $\mathscr{F}$, the inequality (4.21) follows immediately from (4.23).

Now we are in a position to show the convergence of the sequence $\{u^{(k)}, v^{(k)}\}$.

THEOREM 3.1. *Let* $\lambda > K$ *be chosen. Then the sequence* $\{U^{(k)}\} = \{u^{(k)}, v^{(k)}\}$

given by (4.19) *converges to a unique solution* $U = (u, v)$ *of the problem* (3.6)–(3.8). *Moreover,*

$$\| U^{(k)} - U \| \leqslant \frac{K}{\lambda - K} \left(\frac{K}{\lambda}\right)^{k-1} \| U^{(1)} - U^{(0)} \|, \qquad k = 1, 2, \ldots \tag{4.24}$$

Proof. Let $W = (w, w') \in \mathscr{L}^2(\Omega)$ be given. Then by Lemma 3.3 with $g = F_1(W)$, $h = F_2(W)$ there exist unique $u \in D(\bar{\mathscr{A}}_1)$, $v \in D(\bar{\mathscr{A}}_2)$ such that $\bar{\mathscr{A}}_1 u = F_1(W)$, $\bar{\mathscr{A}}_2 v = F_2(W)$. Let $U = (u, v)$ so that $\mathscr{A} U = \mathscr{F}(W)$. In view of Lemma 3.2 we have $U = \mathscr{A}^{-1}\mathscr{F}(W)$. Since this is true for every $W \in \mathscr{L}^2(\Omega)$, the composite operator $\mathscr{A}^{-1}\mathscr{F}$ is everywhere defined on $\mathscr{L}^2(\Omega)$. Furthermore, Lemma 3.2 and the inequality (4.21) imply that for any W, $W' \in \mathscr{L}^2(\Omega)$,

$$\| \mathscr{A}^{-1}\mathscr{F}(W) - \mathscr{A}^{-1}\mathscr{F}(W')\| \leqslant \lambda^{-1} \| \mathscr{F}(W) - \mathscr{F}(W')\| \leqslant \lambda^{-1} K \| W - W' \| . \tag{4.25}$$

It follows from the choice of $\lambda > K$ that $\mathscr{A}^{-1}\mathscr{F}$ is a contraction mapping on $\mathscr{L}^2(\Omega)$ with a contraction constant (K/λ). By the contraction property of $\mathscr{A}^{-1}\mathscr{F}$, the sequence $\{U^{(k)}\}$ determined successively from the equation

$$U^{(k)} = \mathscr{A}^{-1}\mathscr{F}(U^{(k-1)}), \qquad k = 1, 2, \ldots, \tag{4.26}$$

with any $U^{(0)} \in \mathscr{L}^2(\Omega)$ converges to a unique solution U of the equation $U = \mathscr{A}^{-1}\mathscr{F}(U)$ and satisfies the error estimate (4.24). This shows that $U \in D(\mathscr{A})$ and $\mathscr{A} U = \mathscr{F}(U)$, that is, $U = (u, v)$ is the unique solution of the problem (3.6)–(3.8). Since (4.26) is equivalent to (3.18) which is the operator equation for (3.9), (3.10), and since the solutions (3.9), (3.10) are given by (4.19) we conclude that the sequence $\{u^{(k)}, v^{(k)}\}$ determined from (4.19) converges to the unique solution (u, v) of (3.6)–(3.8). This completes the proof of the theorem.

To solve our original problem (2.1)–(2.3) we let $x^{(k)} = e^{\lambda t} u^{(k)}$, $y^{(k)} = e^{\lambda t} v^{(k)}$. Then the recursion formula (4.19) reduces to (3.3). Since the convergence of $\{u^{(k)}, v^{(k)}\}$ implies the convergence of $\{x^{(k)}, y^{(k)}\}$ and $x^{(k)} \to e^{\lambda t} u$, $y^{(k)} \to e^{\lambda t} v$ as $k \to \infty$ we see that the pair $x = e^{\lambda t} u$, $y = e^{\lambda t} v$ is the unique solution of (2.1)–(2.3). In conclusion, we have

THEOREM 3.2. *The sequence* $\{x^{(k)}, y^{(k)}\}$ *given by* (3.3) *converges to a unique solution* (x, y) *of the problem* (2.1)–(2.3).

In the special case of constant matrices A, B, C, D and without external sources the recursion formula (3.3) is reduced to the form

$$\left.\begin{aligned} x^{(k)}(r,t) &= \hat{\phi}(r+t) + A\int_0^t x^{(k-1)}(r+t-\tau,\tau)\,d\tau \\ &\quad + D\int_0^t y^{(k-1)}(r+t-\tau,\tau)\,d\tau \\ y^{(k)}(r,t) &= \hat{\psi}(r-t) + B\int_0^t x^{(k-1)}(r-t+\tau,\tau)\,d\tau \\ &\quad + C\int_0^t y^{(k-1)}(r-t+\tau,\tau)\,d\tau \end{aligned}\right\} k = 1, 2,\ldots \tag{4.27}$$

In particular, if $\phi = \psi = 0$ we have the following result for the problem (1.1)–(1.3).

COROLLARY. *The sequence* $\{x^{(k)}, y^{(k)}\}$ *given by* (4.27) *converges to a unique solution* (x, y) *of the problem* (1.1)–(1.3).

Remark. The above corollary gives a direct method for the determination of the solution of the problem considered in [1] by Bellman.

4. The Well-Posed Problem

In the previous section it is shown that the problem (2.1)–(2.3) has a unique solution which can be constructed by successive integrations of the recursion formula (3.3). In this section we show that this solution depends continuously on the external source p, q, the initial data ϕ, ψ and the boundary data c, d. We recall that every function $w(r, t)$ in $L^2(\Omega)$ may be considered as a vector-valued function $w(t)$ with values in $L^2(0, l)$, where the inner product of any pair $u(r)$, $v(r)$ in $L^2(0, l)$ is given by

$$\langle u, v\rangle_0 = \int_0^l u(r)\cdot v(r)\,dr = \int_0^l \sum_{i=1}^n u_i(r)\,v_i(r)\,dr.$$

For $U(r) = (u(r), u'(r))$ with $u, u' \in L^2(0, l)$ we write $\| U \|_0 = \| u \|_0 + \| u' \|_0$, where $\| u \|_0 = \langle u, u\rangle_0^{1/2}$.

Let $\lambda_j(r, t)$, $j = 1, 2$, be the largest eigenvalue of the respective matrices $\frac{1}{2}(A^T + A)$, $\frac{1}{2}(B^T + B)$ and let $\alpha_j(t)$, $j = 1,\ldots, 4$, be any continuous functions satisfying

$$\begin{aligned} \alpha_j(t) &\geqslant \operatorname*{ess-sup}_{0\leqslant r\leqslant l}(\lambda_j(r,t)), \qquad && j = 1, 2, \\ \alpha_j(t) &\geqslant [\operatorname*{ess-sup}_{0\leqslant r\leqslant l}(\Lambda_j(r,t))]^{1/2}, && j = 3, 4, \end{aligned} \tag{5.1}$$

where $\Lambda_j(t) \geqslant 0$ is defined in Section 3. Set

$$\omega(t) = \max\{\alpha_1(t) + \alpha_3(t), \alpha_2(t) + \alpha_4(t)\} \qquad (t \geqslant 0). \tag{5.2}$$

Then we have the following estimate for the solution (x, y) when the boundary data vanishes.

THEOREM 4.1. *Let $Z = (x, y)$ be the solution of the problem* (2.1)–(2.3) *for the case $c = d = 0$ and let $Q = (p, q)$, $\Phi = (\phi, \psi)$. Then*

$$\| Z(t)\|_0 \leqslant \left(\exp \int_0^t \omega(s)\, ds\right) \| \Phi \|_0 + \int_0^t \left(\exp \int_\tau^t \omega(s)\, ds\right) \| Q(\tau)\|_0\, d\tau \qquad (t \geqslant 0). \tag{5.3}$$

In particular, the solution (x, y) depends continuously on p, q, ϕ and ψ.

Proof. It is easily seen that $(d/dt)\,(\| x(t)\|_0^2)$ exists and

$$\frac{d}{dt}\,(\| x(t)\|_0^2) = 2\langle x_t(t), x(t)\rangle_0 \qquad (t > 0). \tag{5.4}$$

Substitution of the first equation in (2.1) for x_t leads to

$$\| x(t)\|_0 \frac{d}{dt}\,(\| x(t)\|_0) = \langle x_r(t) + A(t)\, x(t) + D(t)\, y(t) + p(t), x(t)\rangle_0 . \tag{5.5}$$

Since by integration and the boundary condition $x(l, t) = c = 0$,

$$2\langle x_r(t), x(t)\rangle_0 = 2 \int_0^l x_r(r, t) \cdot x(r, t)\, dr = - \mid x(0, t)|^2 \leqslant 0; \tag{5.6}$$

also, by the definition of $\alpha_j(t)$,

$$\begin{aligned} 2\langle Ax, x\rangle_0 &= \langle (A^T + A)\, x, x\rangle_0 \leqslant 2\alpha_1(t)\, \| x(t)\|_0^2 , \\ |\langle Dy, x\rangle_0 \,| &\leqslant \| Dy \|_0 \| x \|_0 \leqslant \alpha_4(t)\, \| x(t)\|_0 \| y(t)\|_0 , \end{aligned} \tag{5.7}$$

we obtain from (5.5)–(5.7) that

$$\| x(t)\|_0 \frac{d}{dt}\,(\| x(t)\|_0) \leqslant [\alpha_1(t)\, \| x(t)\|_0 + \alpha_4(t)\, \| y(t)\|_0 + \| p(t)\|_0]\, \| x(t)\|_0 . \tag{5.8}$$

Now if $\| x(t)\|_0 \neq 0$ we can divide (5.8) by $\| x(t)\|_0$ to obtain

$$\frac{d}{dt}\,(\| x(t)\|_0) \leqslant \alpha_1(t)\, \| x(t)\|_0 + \alpha_4(t)\, \| y(t)\|_0 + \| p(t)\|_0 . \tag{5.9}$$

If $\| x(t)\|_0 = 0$ and t is a cluster point then $(d/dt)\,(\| x(t)\|_0) = 0$ so that (5.9) remains valid. Since there are atmost a countable number of isolated points at which $\| x(t)\|_0 = 0$, (5.9) holds for almost all values of t. By the same treatment for $y(t)$ we have

$$\frac{d}{dt}(\| y(t)\|_0) \leqslant \alpha_2(t) \| y(t)\|_0 + \alpha_3(t) \| x(t)\|_0 + \| q(t)\|_0 , \tag{5.10}$$

for almost all t. Addition of (5.9) and (5.10) and using the definition of $\omega(t)$ lead to

$$\frac{d}{dt}(\| Z(t)\|_0) \leqslant \omega(t) \| Z(t)\|_0 + \| Q(t)\|_0 \qquad \text{for almost all } t. \tag{5.11}$$

It follows by integrating (5.11) from 0 to t and using the initial condition (2.3) we obtain the result (5.3). Since $\exp(\int_0^t \omega(s)\, ds)$ is bounded for $t \in [0, T]$ the continuous dependence of (x, y) on p, q, ϕ, ψ follows directly from (5.3). This completes the proof of the theorem.

The result in Theorem 4.1 insures the continuous dependence of the solution (x, y) on data when $c = d = 0$. In case, c, d are nonzero, the transformation $x \to x - c$, $y \to y - d$ implies that (5.3) holds for the function $Z(t) = (x(t) - c, y(t) - d)$ with p, q, ϕ, ψ replaced by $p + Ac + Dd$, $q + Bc + Cd$, $\phi - c$, $\psi - d$, respectively. With these functions in (5.3) we conclude that the solution (x, y) of (2.1)–(2.3) depends continuously on p, q, ϕ, ψ as well as on c, d. This fact together with the existence and uniqueness of a solution given in Theorem 3.2 leads to the following conclusion.

THEOREM 4.2. *The problem* (2.1)–(2.3) *is well posed.*

It is to be noted that the result in Theorem 4.1 insures that $x(r, t)$, $y(r, t)$ increases no faster than exponential order if the sources $p(r, t)$, $q(r, t)$ are of exponential order or less. This fact is consistent with the use of Laplace transform to the problem (1.1)–(1.3) given in [1]. On the other hand, the inequality (5.3) can be used for the study of the stability problem of an equilibrium solution (x_e , y_e). To see this, we let $\tilde{x} = x - x_e$, $\tilde{y} = y - y_e$, where (x, y) is any solution of (2.1)–(2.3). Then the function $\tilde{Z} = (\tilde{x}, \tilde{y})$ satisfies the system (2.1)–(2.3) with $p = q = c = d = 0$, $\tilde{x}(r, 0) = \phi(r) - x_e(r)$, $\tilde{y}(r, 0) = \psi(r) - y_e(r)$. Application of the inequality (5.3) yields

$$\| Z(t)\|_0 \leqslant \exp\left(\int_0^t \omega(s)\, ds\right) \| Z(0)\|_0 \qquad (t \geqslant 0). \tag{5.12}$$

Hence if $\exp(\int_0^t \omega(s)\, ds)$ is uniformly bounded on $[0, \infty)$ then (x_e , y_e) is stable, and if, in addition, $\int_0^t \omega(s)\, ds \to -\infty$ as $t \to \infty$, it is asymptotically stable.

5. Concluding Remarks

In the discussion of the preceeding sections we have given an analytic treatment for the existence and uniqueness of a solution as well as its continuous dependence on the external source and the initial-boundary data. The existence problem is based on the method of successive approximations and the contraction mapping theorem. A significant aspect of this method is that it leads to a recursion formula for the calculation of approximate solutions. Since this formula involves only integrations of known functions the calculation of these approximations is quite straightforward and can be carried out by using a computer. When the recursion formula (4.19) for the transformed problem (3.6)–(3.8) is used, an error estimate for the approximations $\{u^{(k)}, v^{(k)}\}$ is given by (4.24). Furthermore, after the selection of $(u^{(0)}, v^{(0)})$ and λ together with the first iteration for $(u^{(1)}, v^{(1)})$ the relation (4.24) gives a more definite error estimate for the succeeding approximations $(u^{(k)}, v^{(k)})$, $k = 2, 3, \ldots$, and this estimate decreases proportional to the kth power of (K/λ). Since the value of K depends only on the matrices A, B, C, D, larger value of λ makes the convergence of the approximations faster than smaller value of λ. This seems to suggest that a very large value of λ be chosen. However, in view of $x^{(k)} = e^{\lambda t} u^{(k)}$, $y^{(k)} = e^{\lambda t} v^{(k)}$, where $(x^{(k)}, y^{(k)})$ is the corresponding approximate solution of (2.1)–(2.3), large value of λ tends to magnify the error estimates in the original problem. Hence the choice of λ should not be too large. On the other hand, since the error estimate for $U^{(k)} = (u^{(k)}, v^{(k)})$ is proportional to $\| U^{(1)} - U^{(0)} \|$ and $U^{(1)}$ depends on $U^{(0)}$ it is obvious that the choice of $U^{(0)}$ plays an important role in the rate of convergence of the approximations even though it is immaterial as far as the convergence is concerned.

In using any one of the recursion formulas (3.3), (4.19), (4.27) the spatial variable r is fixed throughout the process of iterations. This property is especially useful if one is interested only in the particles density at a specific position such as $x(0, t)$ or $y(l, t)$. It also gives a similarity between the initial boundary value problem (2.1)–(2.3) and a corresponding Cauchy problem in which the spacial variable r is considered as a parameter in the process of successive approximations. This similarity indicates that our approach has, in a sense, the same significance as the approach of invariant imbedding.

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